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Propagation of internal gravity waves in perfectly conducting fluids with shear flow, rotation and transverse magnetic field

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The propagation of internal Alfvén–inertio-gravitational waves in a Boussinesq inviscid adiabatic perfectly conducting shear flow with rotation is investigated in the presence of a transverse magnetic field. It is shown that the effect of the rotational nature of electromagnetic force and Coriolis force is that linear momentum is not conserved anywhere in the fluid even at critical levels, whereas the angular momentum flux is conserved everywhere in the fluid except at the critical levels at which the Doppler-shifted frequency $\Omega_d = 0, \pm\Omega_A$, or $\pm\Omega \pm (\Omega^2+\Omega_A^2)^{\frac{1}{2}}$, where Ω_A is the Alfvén frequency and Ω is the Coriolis frequency, and the angular momentum is transferred to the mean flow there by Alfvén–inertio-gravitational waves. Asymptotic solutions to the wave equation are obtained near the critical levels and it is shown that the effect of the Lorentz force on the waves at the critical levels is to increase the process of critical layer absorption. The condition for neglection of rotation for higher frequency waves is also obtained and is found to be the same in both hydrodynamic and hydromagnetic flows.

1. Introduction

The theory of momentum transport by waves is an area of considerable activity and an examination of specific cases will serve as a check on general theory. Recently Booker & Bretherton (1967) have studied theoretically the propagation of internal gravity waves of small amplitudes in a Boussinesq inviscid adiabatic liquid in a non-rotating system in which the mean horizontal velocity $U_0(z)$ depends on height z only. Assuming that the hydrodynamic Richardson number J_H is everywhere greater than one quarter, they have shown that the waves are attenuated by a factor $\exp\{-2\pi(J_H-\frac{1}{4})^{\frac{1}{2}}\}$ as they pass through the critical level at which the Doppler-shifted frequency is zero. The propagation of groups of internal gravity waves in a shear flow without rotation has been investigated by Bretherton (1966), who used the WKB approximation and showed that a wave packet can never pass through a critical level ($\Omega_d = 0$) nor can it be reflected there. Bretherton (1969) has also investigated the momentum transport by gravity waves in a horizontally uniform shear flow, with both rotating and non-rotating systems, and has shown that an upward transport of horizontal momentum inevitably accompanies the generation of such waves in the atmosphere, the mean flow being affected only at precisely those levels where the waves are dissipated. He has also shown that if the mean wind depends on horizontal position there may be a continuous transfer of momentum from waves to the mean flow during propagation.

Recently Jones (1967) has analysed the propagation of internal gravity waves in a shear flow with rotation. He concludes, using a Boussinesq inviscid adiabatic fluid, that the vertical transport of angular momentum is conserved everywhere, is discontinuous across the critical levels at which the Doppler-shifted frequency is equal to plus or minus the Coriolis frequency, and is continuous across the critical level at which the Doppler-shifted frequency is zero. Jones has also found that the solutions of the rotating system approach asymptotically that of the non-rotating system at distances sufficiently far on either side of the critical levels, even though their behaviour in the vicinity of such levels is quite dissimilar.

The internal gravity waves in a conducting stratified fluid with or without rotation have not been given much attention. This problem is of geophysical interest, particularly in the study of the Earth's core. The atmosphere in the presence of the solar magnetic field is another example of such a fluid. Another topic of geophysical interest is the propagation of internal gravity waves from the troposphere to the ionosphere in the presence of a magnetic field and rotation. This problem is investigated in this paper. The problem of internal gravity waves in a non-rotating system in the presence of a magnetic field is presented elsewhere. Work is in progress to include the effects of viscosity and finite electrical conductivity on the propagation of internal gravity waves, with and without rotation.

The importance of vertical variations in the mean horizontal velocity $U_0(z)$ in the presence of a magnetic field and rotation is measured locally by the total Richardson number

$$J = \frac{N^2+\Omega_A^2+\Omega^2}{(dU_0/dz)^2},$$

which is the algebraic sum of the hydrodynamic, hydromagnetic and rotational Richardson numbers, where

$$N = \left[\left(-\frac{g}{\rho_0}\right)\frac{\partial \rho_0}{\partial z}\right]^{\frac{1}{2}}$$

is the Brunt–Väisälä frequency, Ω_A the Alfvén frequency and Ω the Coriolis frequency. Recently Rudraiah & Narasimha Murthy (1971) have shown that, if J is everywhere greater than one quarter, the flow is stable for small disturbances. We shall confine our attention to such stable situations only.

If the basic horizontal velocity is zero, small disturbances take the form of internal Alfvén–inertio-gravitational waves in a flow with rotation and magnetic field, in which there is an oscillatory interchange between disturbance kinetic and magnetic energies and gravitational potential energy associated with deformation of the surfaces of constant density. The properties of such waves are described succinctly by Chandrasekhar (1961, p. 197). If U_0 varies with z, the disturbance is modified by the shear but, if $J > \frac{1}{4}$, the energy interchange is still of the same type as for internal gravity waves in the presence of rotation and magnetic field.

In this paper we discuss the propagation of internal gravity waves in a conducting fluid with rotation and a transverse uniform applied magnetic field. In particular, we draw attention to a mechanism whereby internal Alfvén–inertio-gravitational waves, once generated, may be reabsorbed by the mean flow without necessarily invoking turbulence or other dissipative processes. The effect of a transverse magnetic field is to increase the singular points of the wave equation and hence to increase the number of singular levels. In fact, singularities of the hydromagnetic rotating system discussed here differ from those of the hydrodynamic system discussed by Jones (1967) both in number and form; there are singular levels at which the Doppler-shifted frequency

$$\Omega_d = 0, \quad \pm\Omega_A, \quad \pm\Omega\pm(\Omega^2+\Omega_A^2)^{\frac{1}{2}}.$$

In other words, the singularities in the hydromagnetic flow are seven in number in comparison with three in hydrodynamic flow (Jones 1967; Bretherton 1969). In addition, we observe that in a conducting rotating flow the vertical transport of angular momentum is conserved, whereas the linear momentum is not conserved everywhere in the fluid.

This study has two objectives. The first is to derive the expression for the angular momentum flux by considering the momentum due to the fields and the material media. Since angular momentum is conserved in a rotating system it is taken as the measure of the magnitude of the wave. The second objective is to show that the solution of the hydromagnetic rotating system approaches the hydromagnetic non-rotating system sufficiently far on either side of the critical levels. These results together support the neglect of rotation for higher frequency internal gravity waves in both hydrodynamic and hydromagnetic flows. The angular momentum flux far away from the critical levels is reduced on either side by a factor $\exp\{-2\pi(J-\frac{1}{4})^{\frac{1}{2}}\}$. Comparing this result with that of the hydrodynamic result of Jones, we see that one of the effects of the Lorentz force is to increase the attenuation of waves at the critical levels.

2. Mathematical formulation

We take axes $Oxyz$ such that Oz is vertical. Suppose that an incompressible heterogeneous inviscid perfectly conducting fluid rotates with an angular velocity $\mathbf{\Omega}$. In a frame of reference rotating with uniform angular velocity $\mathbf{\Omega}$ about the z axis the basic equations are (see Chandrasekhar 1961, p. 197)

$$\begin{bmatrix} \rho_1 D & -2\rho_1\Omega & 0 & D_1 & 0 & -\mu\mathscr{D} & 0 & 0 \\ 2\rho_1 D & \rho_1 D & 0 & D_2 & 0 & 0 & -\mu\mathscr{D} & 0 \\ 0 & 0 & \rho_1 D & D_3 & g & 0 & 0 & -\mu\mathscr{D} \\ 0 & 0 & 0 & 0 & D & 0 & 0 & 0 \\ D_1 & D_2 & D_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D_1 & D_2 & D_3 \\ -\mathscr{D} & 0 & 0 & 0 & 0 & D & 0 & 0 \\ 0 & -\mathscr{D} & 0 & 0 & 0 & 0 & D & 0 \\ 0 & 0 & -\mathscr{D} & 0 & 0 & 0 & 0 & D \end{bmatrix} \begin{bmatrix} q_x \\ q_y \\ q_z \\ P_1 \\ \rho_1 \\ H_x \\ H_y \\ H_z \end{bmatrix} = 0, \quad (1)$$

where
$$D = \partial/\partial t + q_x D_1 + q_y D_2 + q_z D_3,$$
$$\mathscr{D} = H_x D_1 + H_y D_2 + H_z D_3,$$
$$D_1 = \partial/\partial x, D_2 = \partial/\partial y, D_3 = \partial/\partial z;$$

(q_x, q_y, q_z) and (H_x, H_y, H_z) are the components of velocity and magnetic field respectively in the increasing directions of (x, y, z), $P_1 = p + \frac{1}{2}\mu H^2$ is the sum of the hydrodynamic and hydromagnetic pressure and ρ_1 is the density of fluid.

The analysis is restricted by the following assumptions.

(*a*) The motion is three-dimensional.

(*b*) It is inviscid, perfectly conducting and adiabatic.

(*c*) The Boussinesq approximation holds. This implies that the density in (1) is replaced by a constant density except for the introduction of a buoyancy force in the vertical direction which is equal to the product of the gravitational acceleration g and the fluctuations of the density at each point and time from its mean value there, $\rho_0(z)$. The stratification of the mean flow may then be described by the Brunt–Väisälä frequency N, where
$$N^2 = g\beta = -(g/\rho_0)\,\partial\rho_0/\partial z. \tag{2}$$

(*d*) The applied uniform magnetic field H_0 is in the y direction, which is transverse to the mean flow. We note that the problem of the propagation of internal gravity waves with an aligned magnetic field is similar to the case of a transverse magnetic field (i.e. in the y direction) except that Ω_A, which equals Al in the case of a transverse magnetic field, is replaced by Ak in the case of an aligned magnetic field, A being the Alfvén velocity and k and l the wavenumbers in the x and y directions respectively. We further note that if the applied magnetic field H_0 is in the z direction the basic horizontal velocity $U_0(z)$ has to be uniform to satisfy the magnetic induction equation (1). Therefore, to consider the propagation of internal gravity waves in a non-uniform mean flow we avoid having the basic magnetic field H_0 in the z direction.

(*e*) The perturbation quantities $q_x = U_0(z) + u$, $q_y = v$, $q_z = w$, $H_x = h_x$, $H_y = H_0 + h_y$, $H_z = h_z$, $\rho_1 = \rho_0 + \rho$, $P_1 = P_0 + P$ are small compared with the basic flow quantities, and the perturbed quantities are of the form
$$(\text{function of } z)\exp i(kx + ly + \sigma t). \tag{3}$$

(*f*) The total modified Richardson number
$$J = \left[\frac{N^2 + \Omega_A^2}{(dU_0/dz)^2}\right]\left(1 + \frac{l^2}{k^2}\right) > \tfrac{1}{4}$$
everywhere in the fluid to ensure stability for small disturbances (Rudraiah 1970).

The mean density in the presence of a transverse magnetic field H_0 is gravitationally stratified and, as a manifestation of the thermal wind equation, shows a variation in the y direction. For geostrophic balance
$$\partial P_0/\partial y = -2\Omega U_0 \rho_0 \tag{4}$$
and for magnetostatic balance
$$\partial P_0/\partial z = -g\rho_0, \tag{5}$$
where
$$P_0 = p_0 + \tfrac{1}{2}\mu H_0^2.$$

From (4) and (5), we get

$$\frac{1}{\rho_0}\frac{\partial \rho_0}{\partial y} = \frac{2\Omega}{g}\frac{dU_0}{dz} - \frac{2\Omega\beta U_0}{g}. \tag{6}$$

The last term of (6) is negligible in the Boussinesq approximation since it is very small compared with the other terms (see Jones 1967).

The perturbation equations of motion, using (3)–(6), are

$$\begin{bmatrix} \rho_0 D_0 & -2\Omega\rho_0 & \rho_0 D_3 U_0 & -\mu H_0 D_2 & 0 & 0 & D_1 & 0 \\ 2\Omega\rho_0 & \rho_0 D_0 & 0 & 0 & -\mu H_0 D_2 & 0 & D_2 & 0 \\ 0 & 0 & \rho_0 D_0 & 0 & 0 & -\mu H_0 D_2 & D_3 & g \\ D_1 & D_2 & D_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & D_2\rho_0 & D_3\rho_0 & 0 & 0 & 0 & 0 & D_0 \\ 0 & 0 & 0 & D_1 & D_2 & D_3 & 0 & 0 \\ -H_0 D_2 & 0 & 0 & D_0 & 0 & -D_3 U_0 & 0 & 0 \\ 0 & -H_0 D_2 & 0 & 0 & D_0 & 0 & 0 & 0 \\ 0 & 0 & -H_0 D_2 & 0 & 0 & D_0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \\ h_x \\ h_y \\ h_z \\ P \\ \rho \end{bmatrix} = 0, \tag{7}$$

where $$D_0 = \partial/\partial t + U_0\,\partial/\partial x.$$

Equations (3)–(7) can be combined to yield the wave equation

$$\left[\frac{4\Omega^2\Omega_d^2}{\Omega_d^2-\Omega_A^2} - (\Omega_d^2-\Omega_A^2)\right]\frac{d^2w}{dz^2} + \left[4i\Omega l - \frac{8\Omega^2 k\Omega_d^3}{(\Omega_d^2-\Omega_A^2)^2} - \frac{2k\Omega_A^2}{\Omega_d}\right]\frac{dU_0}{dz}\frac{dw}{dz}$$
$$+\left[(\Omega_d^2-\Omega_A^2-N^2)(k^2+l^2) - \left(\frac{4ik\Omega l}{\Omega_d} - \frac{2k^2\Omega_A^2}{\Omega_d^2}\right)\left(\frac{dU_0}{dz}\right)^2 + \left\{\frac{k}{\Omega_d}(\Omega_d^2-\Omega_A^2) + 2i\Omega l\right\}\frac{d^2U_0}{dz^2}\right] w = 0, \tag{8}$$

where $A = (\mu H_0^2/\rho_0)^{\frac{1}{2}}$ is the Alfvén velocity, $\Omega_A = Al$ is the Alfvén frequency, $\Omega_d = kU_0 + \sigma$ is the Doppler-shifted frequency and $N = (g\beta)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency.

In the absence of rotation and Lorentz force, (8) is the wave equation for internal gravity waves discussed by Booker & Bretherton (1967). In the absence of the Lorentz force, i.e. $\Omega_A = 0$, (8) reduces to the hydrodynamic wave equation of Jones (1967) and in this case the interaction of rotation and gravity generates inertio-gravitational waves. When the Lorentz force is included, however, Alfvén waves are generated and are coupled with the inertio-gravitational waves governed by (8) and the resulting waves are calledAlfvén–inertio-gravitational waves.

Equation (8) is singular at

$$\Omega_d = 0, \quad \pm\Omega_A, \quad \pm\Omega \pm (\Omega^2+\Omega_A^2)^{\frac{1}{2}}. \tag{9}$$

These points are shown in figure 1. From this it follows that compared with three singular levels at $\Omega_d = 0$, $\pm 2\Omega$ in hydrodynamic flow, there are seven singular levels in hydromagnetic flow given by (9). These critical levels are nothing but resonance levels where there will be maximum absorption of energy. Thus, one of the effects of a Lorentz force coupled with rotation and gravity is to generate Alfvén–inertio-gravitational waves which act as a source of high-power magnetohydrodynamic resonance in the region from troposphere to ionosphere. We also note that the effect of the Lorentz force on the flow is to increase the number of critical levels and also to push the corresponding hydrodynamic critical levels ($\Omega_d = 0, \pm 2\Omega$) to great heights on either side of the critical level $\Omega_d = 0$. Other wave parameters are related to w by

$$\begin{bmatrix} u \\ v \\ \xi \\ \eta \\ h_x \\ h_y \\ h_z \\ P \\ \rho \end{bmatrix} = \frac{i}{(k^2+l^2)\,\Omega_d^2} \begin{bmatrix} \Omega_d l^2 & \Omega_d \alpha_1 \\ -\Omega_d kl & \Omega_d \alpha_2 \\ -il^2 & -i\alpha_1 \\ ikl & -i\alpha_2 \\ -H_0 k^2 l & H_0 l \alpha_1 \\ -H_0 k l^2 & H_0 l \alpha_2 \\ \dfrac{-i\Omega_d H_0 l(k^2+l^2)}{(dU_0/dz)} & 0 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \dfrac{2i\rho_0\Omega\alpha_2}{g}\dfrac{dU_0}{dz} \end{bmatrix} \begin{bmatrix} w\,dU_0/dz \\ dw/dz \end{bmatrix}, \tag{10}$$

where

$$\alpha_0 = \Omega_d^2 - \Omega_A^2, \quad \alpha_1 = \Omega_d(k - 2il\Omega\Omega_d/\alpha_0), \quad \alpha_2 = \Omega_d(l + 2ik\Omega\Omega_d/\alpha_0),$$

$$\alpha_3 = \rho_0\{\alpha_0 k + 2il\Omega\Omega_d\}, \quad \alpha_4 = -\rho_0\Omega_d\{\alpha_0 - 4\Omega^2\Omega_d^2/\alpha_0\},$$

$$\alpha_5 = -\frac{\rho_0}{q\,dU_0/dz}\left\{2ikl\Omega\left(\frac{dU_0}{dz}\right)^2 + N^2\Omega_d(k^2+l^2)\right\}.$$

Here ξ and η are the x and y displacements of a fluid parcel from its rest position.

3. Angular momentum flux

In the case of non-rotating hydrodynamic flow Bretherton (1969) has pointed out that when a stably stratified airstream flows over irregular topography, gravity waves are excited and propagate upwards, transferring energy and linear momentum possibly to great heights. In the corresponding problem of flow with rotation, Jones (1967) has concluded that the linear momentum is not conserved but instead the angular momentum is conserved and hence is a measure of the intensity of waves. Thus in this case the inertio-gravitational waves propagate upwards, transferring energy and angular momentum possibly to great heights. Recently Bretherton (1969) has re-examined the inertio-gravitational waves and concluded that we can regard the angular momentum flux considered by Jones (1967) as the vertical momentum flux and draw the same conclusions as in the non-rotating case. However, in the case of hydromagnetic rotating flows, in addition to the transfer of momentum by inertio-gravitational waves there will be a transfer of momentum by Alfvén waves. In other words, the transfer of energy and angular momentum is by Alfvén–inertio-gravitational waves. We note that the principle of conservation of momentum is valid only when the momentum due to magnetic field is taken into account along with that of the matter which produces it. Thus the total momentum flux will be the algebraic sum of the wave momentum fluxes in the fields and the material media. To obtain an expression for the momentum flux we consider the mean position of a fluid parcel at (x_0, y_0, z_0) and its instantaneous horizontal displacement from this position given by ξ and η from (10). Since the basic equations of motion are independent of the choice of the axis of rotation, we assume that the axis of rotation is at the origin. Then the instantaneous angular momentum per unit volume of the fluid parcel, using the linearized approximation, is

$$[(x_0+\xi)^2+(y_0+\eta)^2]\,\Omega+(x_0+\xi)\,v-(y_0+\eta)\,(U_0+u)$$
$$\approx (x_0^2+y_0^2)\,\Omega-y_0U_0+x_0(v+2\Omega\xi)-y_0(u-2\Omega\eta), \quad (11)$$

where we have assumed that $U_0 \ll 2\Omega y_0$ (i.e. rapid rotation).

To determine the angular momentum flux due to the electromagnetic field we make use of the electromagnetic stress tensor

$$T_{\alpha\beta} = E_\alpha D_\beta + \beta_\alpha H_\beta - \tfrac{1}{2}(\mathbf{E}.\mathbf{D}+\mathbf{B}.\mathbf{H})\,\delta_{\alpha\beta},$$

where $-T_{\alpha\beta}$ may be interpreted as the momentum flux tensor (Stratton 1941) and $\delta_{\alpha\beta}$ is the Kronecker delta. Thus the vertical flux of angular momentum due to the electromagnetic field using the linearized approximation, is

$$-T_{yz}(x_0+\xi)+T_{xz}(y_0+\eta) \approx -\mu h_y h_z x_0+\mu h_x h_z y_0. \quad (12)$$

Then the resultant vertical transport of angular momentum from (11) and (12), taking the time average, is

$$M = x_0\{\rho_0\overline{(v+2\Omega\xi)\,w}-\mu\overline{h_y h_z}\}-y_0\{\rho_0\overline{(u-2\Omega\eta)\,w}-\mu\overline{h_x h_z}\}$$
$$= \tfrac{1}{2}x_0\,\mathrm{Re}\,\{\rho_0(v+2\Omega\xi)\,w^*-\mu h_y h_z^*\}-\tfrac{1}{2}y_0\,\mathrm{Re}\,\{\rho_0(u-2\Omega\eta)\,w^*-\mu h_x h_z^*\}, \quad (13)$$

where w^* and h_z^* are the complex conjugates of w and h_z respectively, Re denotes the real part and the overbar denotes the time average. Comparing the angular momentum flux given by (13) with that of the hydrodynamic result of Jones (1967), we find that the effect of the Lorentz force on the flow is to reduce the flux of angular momentum. This is consistent with the particle motion being retarded by the Lorentz force.

From (13), using (10), we have

$$\mathrm{Re}\,[\rho_0(v+2\Omega\xi)\,w^*-\mu h_y h_z^*] = (l/(k^2+l^2))\,G \quad (14)$$

and
$$\mathrm{Re}\,(\rho_0(u-2\Omega\eta)\,w^*-\mu h_x h_z^*] = (k/(k^2+l^2)\,G, \quad (15)$$

where

$$G = \mathrm{Re}\left\{\rho_0\left[\frac{2\Omega l}{\Omega_d^2}\frac{dU_0}{dz}ww^*-\frac{i(4\Omega^2-\Omega_d^2+\Omega_A^2)}{\Omega_d^2-\Omega_A^2}w^*.\frac{dw}{dz}-\frac{i\Omega_A^2}{\Omega_d^2}w^*\frac{dw}{dz}\right]\right\}. \quad (16)$$

Differentiating (16) with respect z, using (8) and assuming k, l, σ, Ω are all real, we find that

$$dG/dz = 0. \quad (17)$$

Thus $\quad \mathrm{Re}\,[\rho_0(v+2\Omega\xi)\,w^*-\mu h_y h_z^*], \quad \mathrm{Re}\,[\rho_0(u-2\Omega\eta)\,w^*-\mu h_x h_z^*],$

and the vertical flux of angular momentum are constant with height. This is true at any level except the critical levels, where substitution of (8) is invalid.

If L denotes the vertical flux of linear momentum then

$$L = \tfrac{1}{2}\operatorname{Re}\,[\rho_0(u+v)\,w^* - \mu(h_x+h_y)\,h_z^*]. \tag{18}$$

Therefore

$$\frac{dL}{dz} = \frac{2\Omega}{k^2+l^2}\operatorname{Re}\left[-\frac{i(k+l)\,(\Omega_d^2-\Omega_A^2)}{2\Omega\Omega_d^2}\,w^*\frac{d^2w}{dz^2} - \frac{i(k+l)\,k\Omega_A^2\,(dU_0/dz)}{\Omega\Omega_d^3}\,w^*\frac{dw}{dz} + \frac{k-l}{\Omega_d}\left(\frac{dw}{dz}\frac{dw^*}{dz} + \frac{d^2w}{dz^2}w^* - \frac{k}{\Omega_d}\frac{dU_0}{dz}\frac{dw}{dz}w^*\right)\right]. \tag{19}$$

Hence the vertical flux of linear momentum is not conserved in hydromagnetic flow, as observed by Jones (1967) for hydrodynamic flow. In the absence of rotation (19) becomes

$$dL/dz = 0.$$

Thus in the case of non-rotating hydromagnetic flow the vertical flux of linear momentum is conserved as observed by Booker & Bretherton (1967). Further, we note that

$$\lim_{\left|\frac{\Omega_d}{2\Omega}\right|\to\infty} \operatorname{Re}\{\rho_0(v+2\Omega\xi)\,w^* - \mu h_y h_z^*\} = \lim_{\left|\frac{\Omega_d}{2\Omega}\right|\to\infty} \operatorname{Re}\,(\rho_0 v w^*) = \frac{l\rho_0}{k^2+l^2}\operatorname{Re}\left(iw^*\frac{dw}{dz}\right), \tag{20}$$

and

$$\lim_{\left|\frac{\Omega_d}{2\Omega}\right|\to\infty} \operatorname{Re}\{\rho_0(u-2\Omega\eta)\,w^* - \mu h_x h_z^*\} = \lim_{\left|\frac{\Omega_d}{2\Omega}\right|\to\infty} \operatorname{Re}\,(\rho_0 u w^*) = \frac{k\rho_0}{k^2+l^2}\operatorname{Re}\left(iw^*\frac{dw}{dz}\right). \tag{21}$$

Hence, far away from the critical levels, the vertical transports of angular and linear momentum tend to the same quantity and the momentum transport due to magnetic field is small compared with that of the material media.

In other words, the transport of momentum near critical levels is by Alfvén–inertio-gravitational waves, whereas the transport of momentum far away from critical levels is mainly by inertio-gravitational waves.

4. Asymptotic solution to the hydromagnetic wave equation away from the critical levels

To find asymptotic solutions of (8) away from its singular points, we make the following two assumptions: (i) The velocity shear dU_0/dz is independent of height; (ii) $\Omega_d \ll N$, i.e. the Doppler-shifted frequency is very much less than the Brunt–Väisälä frequency. Without any loss of generality we can take $z = 0$ at the height at which $\Omega_d = 0$ and write (8) as

$$\left[\frac{4\Omega^2(k\,dU_0/dz)^2 z^2}{(k\,dU_0/dz)^2 z^2 - \Omega_A^2} - \left\{\left(k\frac{dU_0}{dz}\right)^2 z^2 - \Omega_A^2\right\}\right]\frac{d^2w}{dz^2} + \left[4i\Omega l - \frac{8k\Omega^2(k\,dU_0/dz)^3 z^3}{\{(k\,dU_0/dz)^2 z^2 - \Omega_A^2\}^2} - \frac{2\Omega_A^2 k}{(k\,dU_0/dz)\,z}\right]\frac{dU_0}{dz}\frac{dw}{dz} - \left[(\Omega_A^2+N^2)\,(k^2+l^2) + \left(\frac{4il\Omega\,dU_0/dz}{z} - \frac{2\Omega_A^2}{z^2}\right)\right]w = 0. \tag{22}$$

To find power-series solutions for (22) in descending powers of z we substitute $\zeta = 1/z$. The resulting equation can be solved in a power-series expansion about $\zeta = 0$ by the method of Frobenius. The expansion is valid in the range

$$\left|\frac{k}{\Omega+(\Omega^2+\Omega_A^2)^{\frac{1}{2}}}\frac{dU_0}{dz}\frac{1}{\zeta}\right| = \left|\frac{k}{\Omega+(\Omega^2+\Omega_A^2)^{\frac{1}{2}}}\frac{dU_0}{dz}z\right| > 1. \tag{23}$$

The resulting expansion, in terms of z, is

$$w = a_0 z^{-a}[1 + a_1 z^{-1} + a_2 z^{-2} + \ldots] + b_0 z^{-b}[1 + b_1 z^{-1} + b_2 z^{-2} + \ldots], \tag{24}$$

where a_0 and b_0 are arbitrary constants of integration,

$$a = -\tfrac{1}{2} - i\mu_m, \quad b = -\tfrac{1}{2} + i\mu_m, \quad \mu_m = (J - \tfrac{1}{4})^{\frac{1}{2}},$$

$$a_1 = b_1 = -\frac{2i\Omega l}{k^2 \, dU_0/dz}, \quad a_2 = [f(\lambda)]_{\lambda=a}, \quad b_2 = [f(\lambda)]_{\lambda=b},$$

$$f(\lambda) = \frac{4J_R}{k^2}\left[\frac{\lambda(\lambda+3)}{2(2\lambda+3)} - \frac{l^2}{k^2}\frac{(\lambda+2)}{(2\lambda+3)}\right] + \frac{J_m}{k^2}\left[\frac{\lambda(3\lambda+5)}{2(2\lambda+3)} + \frac{J+1}{2\lambda+3}\right],$$

$J_R = \dfrac{\Omega^2}{(dU_0/dz)^2}$ is the rotational Richardson number,

$J_m = \dfrac{\Omega_A^2}{(dU_0/dz)^2}$ is the magnetic Richardson number.

When J is moderately large (i.e. $J \geqslant 1$) the coefficients a and b are in general of magnitude unity or a little larger. Therefore, if z is such that

$$|\Omega_d| = \left|k\frac{dU_0}{dz}z\right| \gg \left|2\Omega\left(1 + \frac{l^2}{k^2}\right)^{\frac{1}{2}}\right| \tag{25}$$

equation (24) is approximately

$$w \approx a_0 z^{-a}[1 + c_1 z^{-2} + \ldots] + b_0 z^{-b}[1 + d_1 z^{-2} + \ldots], \tag{26}$$

where

$$c_1 = \frac{J_m}{k^2}\left[\frac{a(3a+5)}{2(2a+3)} + \frac{J+1}{2a+3}\right],$$

$$d_1 = \frac{J_m}{k^2}\left[\frac{b(3b+5)}{2(2b+3)} + \frac{J+1}{2b+3}\right].$$

However, we note that if in addition to approximation (25) z is such that

$$\left|k\frac{dU_0}{dz}z\right| \gg \left|\Omega_A\left(1 + \frac{l^2}{k^2}\right)^{\frac{1}{2}}\right| \tag{27}$$

equation (26) is approximately

$$w \approx a_0 z^{\frac{1}{2} + i\mu_m} + b_0 z^{\frac{1}{2} - i\mu_m} \tag{28}$$

and reduces to the Booker & Bretherton (1967) solution when $\Omega_A = 0$.

In the absence of rotation (22) takes the form

$$\left[\left(k\frac{dU_0}{dz}\right)^2 z^2 - \Omega_A^2\right]\frac{d^2w}{dz^2} + \frac{2\Omega_A^2}{z}\frac{dw}{dz} + \left[(N^2 + \Omega_A^2)(k^2 + l^2) - \frac{2\Omega_A^2}{z^2}\right]w = 0, \tag{29}$$

and its solution is

$$w = c_0 z^{-a}[1 + c_1 z^{-2} + \ldots] + d_0 z^{-b}[1 + d_1 z^{-2} + \ldots], \tag{30}$$

where c_0 and d_0 are constants of integration.

If the Alfvén frequency is small (29) takes the form

$$z^2\, d^2w/dz^2 + Jw = 0 \tag{31}$$

and its solution is

$$w = c_0' z^{\frac{1}{2}+i\mu_m} + d_0' z^{\frac{1}{2}-i\mu_m}, \tag{32}$$

where c_0' and d_0' are constants of integration. Since (26) is similar to (30), (25) is a good approximation when neglecting rotation far away from critical levels. This approximation is the same as the hydrodynamic approximation (Jones 1967) which neglects rotation.

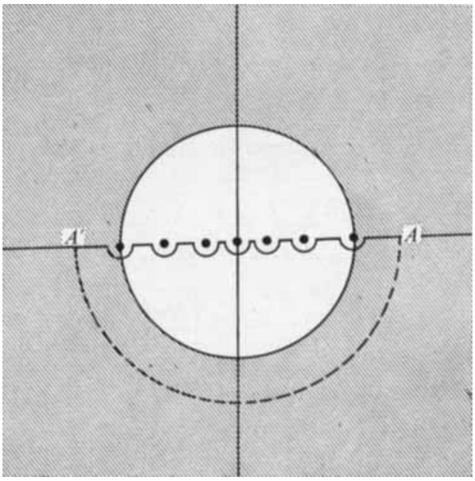

FIGURE 1. Paths of integration in complex z plane around the singularities (●) of (22). ▧, region of validity of asymptotic solution given by (24).

The constants of integration a_0 and b_0 in (24) can be determined by prescribing the boundary conditions at point A, say of figure 1. If we can follow a path, such as the dotted line, through the (shaded) region of validity of (24) in the complex z plane we can use this equation to determine w at A'. To the extent that (26) is a valid approximation to (24) along this path, that is to the extent that the inequality (25) holds, solutions at A with and without rotation will be identical. This follows from the similarity of (26) and (30). Inequality (25) can be met increasingly well as A and A' are moved away from the singularities.

In integrating (22), (29) and (31) through singularities it is essential to know whether to pass above or below the singularities in the complex plane. Following the arguments of Jones (1967) the integration is carried below the singularities. This is justifiable on the basis that his Rayleigh damping equations and our hydromagnetic equations are similar. We note, however, that a full justification for selecting the path of integration can be deduced from an initial-value problem similar to that of Booker & Bretherton (1967). The path of integration is shown by the solid line A–A' of figure 1. As the integration path lies below the singularities we have

$$z \approx |z|e^{-i\pi} \quad (z < 0). \tag{33}$$

5. Asymptotic solutions to the hydromagnetic wave equation near the critical levels

In this section we seek solutions of (8) near the critical levels and interpret their behaviour through a wave-group approach. If $U_0(z), N$, Ω and Ω_A do not vary very much over a wavelength, an Alfvén–inertio-gravitational wave with horizontal wavenumbers k and l and vertical wavenumber m satisfies a dispersion relation

$$\Omega_d^2 = \frac{1}{2(k^2+l^2+m^2)}\{N^2(k^2+l^2)+4\Omega^2m^2+2\Omega_A^2(k^2+l^2+m^2) + [(4\Omega^2m^2+N^2(k^2+l^2))^2+16\Omega^2\Omega_A^2m^2(k^2+l^2+m^2)]^{\frac{1}{2}}\}. \quad (34)$$

From this it follows that as the wave approaches the critical level

$$\Omega_d = \Omega+(\Omega^2+\Omega_A^2)^{\frac{1}{2}}, \quad m \to \infty; \quad (35)$$

so that the wave fronts become more and more horizontal and hence the kinetic and magnetic energies are entirely in the horizontal motion, whereas the potential energy is still associated with vertical displacements. In other words, as the wave approaches this level it becomes an inertial oscillation. The wave then ceases to propagate relative to the local basic flow and the vertical group velocity

$$\frac{\partial \sigma}{\partial m} = \frac{m[(\Omega_d^2-\Omega_A^2)^2-4\Omega^2\Omega_d^2]}{\Omega_d[N^2(k^2+l^2)+4m^2\Omega^2-2(m^2+k^2+l^2)(\Omega_d^2-\Omega_A^2)]} \quad (36)$$

tends to zero as the wave approaches the critical level ($z = z_u$) given by (35). Near the level z_u at which $\Omega_d = \Omega+(\Omega^2+\Omega_A^2)^{\frac{1}{2}}$

$$\frac{dz}{dt} = \frac{k_1(z-z_u)^{\frac{3}{2}}}{k_2(z-z_u)+k_3}, \quad (37)$$

where k_1, k_2 and k_3 are all constants. This may be integrated to give

$$k_2(z-z_u)^{\frac{1}{2}}-k_3/(z-z_u)^{\frac{1}{2}} = \tfrac{1}{2}k_1 t; \quad (38)$$

it takes an infinite time for the wave packet actually to reach this critical level. Thus the wave is neither transmitted nor reflected and simply slows down until either viscosity, magnetic viscosity, turbulence or other nonlinearities destroy it. Similar results follow at the lower critical level $\Omega_d = -\Omega-(\Omega^2+\Omega_A^2)^{\frac{1}{2}}$. If

$$|\Omega_d| < \Omega+(\Omega^2+\Omega_A^2)^{\frac{1}{2}},$$

m becomes imaginary and this zone will be a 'forbidden zone' for propagation of waves.

To discover the limitations of the wave-group approach we consider the momentum transport across the critical levels, which depends on the solution of wave equation (8) near the critical levels. If ξ_1 is the vertical distance from the height at which $\Omega_d = 0$ then in this neighbourhood the solution of (8) is

$$w = \xi_1[c_5+c_6\log\xi_1][1+O(\xi_1)]+c_1\xi_1^2[1+O(\xi_1)]. \quad (39)$$

Substituting this value of w into G of equation (17) and neglecting higher powers of ξ_1, we find that the leading term in a power series for G is not a constant but is discontinuous across $\Omega_d = 0$. In this case G and the angular momentum transport are discontinuous here and hence the waves are absorbed at this critical level. This is contrary to the hydrodynamic result (Jones 1967), namely that the angular momentum transport is continuous across the critical level $\Omega_d = 0$. The reason why the angular momentum transport is not conserved across the critical level $\Omega_d = 0$ is mainly the non-conservative nature of the Lorentz force $\mathbf{J} \times \mathbf{B}$. In other words, the electromagnetic force $(\mathbf{J} \times \mathbf{B})$ produces an additional torque on the rotating conducting fluid across the critical level $\Omega_d = 0$ which was absent in hydrodynamic flow

If instead ξ_2 is the distance to the level at which $\Omega_d = \Omega_A$, the series solution in this neighbourhood is

$$w = c_8[1 + O(\xi_2)] + c_9 \log \xi_2[1 + O(\xi_2)]. \tag{40}$$

Substituting this expression in (16), we find that in general the angular momentum transport is also discontinuous across this critical level. A similar result holds for the other critical levels at $\Omega_d = -\Omega_A, \pm\Omega \pm (\Omega^2 + \Omega_A^2)^{\frac{1}{2}}$.

The above angular momentum analysis shows that the waves are absorbed at all the critical levels, whereas the wave-group analysis shows that the waves are absorbed only at the uppermost and the lowest critical levels, namely

$$\Omega_d = \pm [\Omega + (\Omega^2 + \Omega_A^2)^{\frac{1}{2}}],$$

and there will be no waves in the forbidden zone.

It is of interest to find the angular momentum flux away from all seven of the critical layers. The solution away from the critical levels, neglecting higher order terms in (24), is

$$w = a_0 z^{\frac{1}{2}+i\mu_m} + b_0 z^{\frac{1}{2}-i\mu_m}. \tag{41}$$

The first solution in (41) represents an upward propagating wave and the second one represents the downward propagating wave (see Booker & Bretherton (1967) for a discussion of this interpretation). If, for the sake of definiteness, we fix the branch of the complex powers in (41) by taking

$$z^{\frac{1}{2}\pm i\mu_m} = |z|^{\frac{1}{2}} \exp\{\mp i\mu_m \log|z|\} \quad \text{for} \quad z > 0, \tag{42}$$

then it follows that

$$z^{\frac{1}{2}\pm i\mu_m} = -i \exp\{\pm \mu_m \pi\} |z|^{\frac{1}{2}} \exp\{\pm i\mu_m \log|z|\} \quad \text{for} \quad z < 0. \tag{43}$$

In this case the expression for G is

$$G \begin{cases} = \rho_0 \mu_m [-|a_0|^2 + |b_0|^2] & \text{when} \quad z > 0, \\ = \rho_0 \mu_m [|a_0|^2 e^{2\mu_m \pi} - |b_0|^2 e^{-2\mu_m \pi}] & \text{when} \quad z < 0. \end{cases} \tag{44}$$

Thus the magnitude of each term in (41) at a given distance from the critical levels is not the same above and below the critical levels but differs by a factor of $\exp\{\pm \mu_m \pi\}$, whereas the angular momentum flux G differs by a factor $\exp\{-2\mu_m \pi\}$. In hydrodynamics, Jones (1967) has shown numerically for fairly small values of the Coriolis frequency Ω that, although the wave structure near a critical level is completely altered, the transmission coefficient is very close to the value given by Booker & Bretherton (1967). The same situation prevails even in hydromagnetic flow except that the transmission coefficient has decreased by a factor of the magnetic Richardson number J_m compared with that of Booker & Bretherton (1967). From this it is clear that the effect of the Lorentz force is to increase the attenuation of the waves across the critical levels. Thus from the above arguments we are lead to the conclusion, as in hydrodynamics (Bretherton 1969), that the process of critical-layer absorption depends only on the gross features of the situation and not on the details of the critical layer.

6. Conclusions

The vertical transport of angular momentum is not conserved when we take only the momentum flux in the material media. It is conserved, however, if we consider the propagation of momentum due to electromagnetic field in addition to the material media. Thus the principle of conservation of momentum is strictly valid only when the momentum of an electromagnetic field is taken into account, along with that of the matter which produces it. Therefore, the effect of magnetic field on a rotating system is to make the vertical flux of angular momentum conserved everywhere in the fluid except at the critical layers and to increase the attenuation across the critical layers. Further, the vertical flux of linear momentum is not conserved throughout the fluid. These results are in agreement with the hydrodynamic results of Jones (1967). However, we find that the flux of angular momentum is discontinuous across the critical level $\Omega_d = 0$ which is contrary to the hydrodynamic result of Jones (1967). Also, the angular momentum flux away from the critical levels differs by a factor $\exp(-2\mu_m \pi)$ on either side of the critical levels. Thus, the effect of the Lorentz force at the critical levels is to increase the attenuation of Alfvén–inertio-gravitational waves. In order to understand the attenuation of waves across the critical levels we have discussed both the wave-group and the angular momentum transport approaches. The wave-group approach suggests that the waves are completely absorbed at the critical levels $\Omega_d = \pm[\Omega + (\Omega^2 + \Omega_A^2)^{\frac{1}{2}}]$, whereas the more accurate angular momentum transport approach shows that the waves are transmitted across the critical levels in spite of being attenuated.

The effect of rotation with linear velocity shear can be ignored in regions where the Doppler-shifted frequency satisfies the condition

$$|\Omega_d| \gg |2\Omega(1 + l^2/k^2)^{\frac{1}{2}}|.$$

In zones where this condition does not hold, the rotating and non-rotating systems show widely differing solutions which, however, converge on either side of the zone. We also note that in addition to this condition

$$|\Omega_d| \gg |\Omega_A(1 + l^2/k^2)^{\frac{1}{2}}|,$$

the waves are similar to those discussed by Booker & Bretherton (1967).

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